KERALA UNIVERSITY

First semester MSc Degree Exmanation

Model Question Paper: Branch - Mathematics

MM-213:Ordinary Differntial Equations

(2020 Admission onwards)

Time: 3 hours **Max. Marks:75**

Part A

Answer any 5 questions from among the questions 1 to 8

Each question carries 3 marks

1. Show that $f(x,y)=xy^2$ satisfies a Lipschitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$
2. What are regular singular points of the differential equation $y'' + P(x)\, y' + Q(x) y = 0$. Find the regular singular points of $(1-x^2)\, y'' - 2x\, y' + p(p+1)y = 0$
3. Show that $(1+x)^p$ is the hyper geometric series $F(-p,b,b,-x)$
4. Write the Rodrigues formula for $P_n(x)$. Also find $P_2(x)$ and $P_3(x)$.
5. Show that $\begin{cases} x = e^{4t} \\ y = e^{4t} \end{cases}$ and $\begin{cases} x = e^{-2t} \\ y = -e^{-2t} \end{cases}$ are solutions of the homogenous system $\begin{cases} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 3x + y \end{cases}$
6. If W(t) is the Wronskian of the two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the homogenous system $\begin{cases} \frac{dx}{dt} = a_1(t) + b_1(t)y \\ \frac{dy}{dt} = a_2(t) + b_2(t)y \end{cases}$, prove that W(t) is either identically zero or nowhere zero on [a,b]
7. Define (a) Saddle point (b) Center , of a nonlinear differential equation.
8. Prove that the function $E(x,y)=ax^2+bxy+cy^2$ is positive definite if and only if $a > 0$ and $b^2-4ac < 0$ and negative definite if and only if $a<0$ and $b^2-4ac < 0$

(5 ✕ 3 = 15)

Part B

Answer all questions from 9 to 13

Each question carries 12 marks

9. A (a) Solve $y' = y^2$, $y(0)=1$, using Picard's method starting with $y_o(x)=1$ and by calculating $y_1(x)$, $y_2(x)$ and $y_3(x)$ (5 marks)

   (b) Let x_o be an ordinary point the differential equation $y''+p(x)y'+Q(x)y=0$ .Show that there exist a unique function y(x) analytic at x_o ,which is a solution of the differential equation satisfying $y(x_o)= a_o$ and $y'(x_o)=a_1$ (7 marks)

OR

B (a) State Picard's theorem and use this to solve a system of first order equations

$$\begin{cases} \frac{dy}{dx} = f(x, y, z), \ y(x_o) = y_o \\ \frac{dz}{dx} = g(x, y, z), \ z(x_o) = x_o \end{cases}$$ (6 marks)

(b) Find the indicial equation and its roots of

$2x^2y'' + x(2x+1)y' - y = 0$ (6 marks)

OR

10. A (a) Consider the Chebyshev's equation $(1-x^2)y'' - xy' + p^2y = 0$,where p is a non negative constant. Transform it to a hyper geometric equatio by replacing x by $t = \frac{(1-x)}{2}$ and hence find its general solution near x=1 (7 marks)

(b) If $\frac{1}{\sqrt{1-2xt+t^2}} = p_o(x) + p_1(x)t + p_2(x)t^2 + \ldots + p_n(x)t^n + \ldots$ show that $p_n(1) = p_n(-1) = (-1)^n$ and $p_{2n+1}(0) = 0$ (5marks)

OR

B (a) Determine the nature of the point x=∞ for the Bessel's equation

$x^2 y'' + x y' + (x^2 - p^2)y = 0$ (6marks)

(b) Prove the Orthogonality properties of Legendre polynomials (6marks)

11. A (a) Solve the Bessel's equation $x^2 y'' + x y' + (x^2 - p^2)y = 0$ and find the Bessel function of the first kind of order p (8marks)

(b) Prove that $\frac{d[x^p J_p(x)]}{dx} = x^p J_{p-1}(x)$ (4marks)

OR

B (a) Define the gamma function Γ(p) and show that $\Gamma(p+1) = p\,\Gamma(p)$ (4marks)

(b) Prove the orthogonality properties of Bessel's function (8marks)

12. A (a) Solve $\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = 4x - 2y \end{cases}$ (6marks)

(b) If the two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ have a Wronskian W(t) that does not vanish on (a,b), then show that $\begin{cases} x = c_1x_1(t) + c_2x_2(t) \\ y = c_1y_1(t) + c_2y_2(t) \end{cases}$ is the general solution of

$$\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$$ (6marks)

OR

B (a) Show that $\begin{cases} x = 3t - 2 \\ y = -2t + 3 \end{cases}$ is a particular solution of the non homogenous system

$$\begin{cases} \frac{dx}{dt} = x + 2y + t - 1 \\ \frac{dy}{dt} = 3x + 2y - 5t - 2 \end{cases}$$. Also write the general solution of this system (8marks)

(b) Find the Wronskian of $x_1(t) = e^t\cos(t)$, $x_2(t) = e^{-t}\cos(t)$, $y_1(t) = e^{-t}\sin(t)$, $y_2(t) = e^t\sin(t)$ (4marks)

13. A (a) Prove that the critical point (0,0) of the linear system $\begin{cases} \frac{dx}{dt} = a_1x + b_1y \\ \frac{dy}{dt} = a_2x + b_2y \end{cases}$ is stable if and only if both roots of the auxiliary equation $m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0$ have non positive real parts , and it is asymptotically stable if and only if both roots have negative real parts (8marks)

(b) Prove that the critical point (0,0) of the linear system $\begin{cases} \frac{dx}{dt} = a_1x + b_1y \\ \frac{dy}{dt} = a_2x + b_2y \end{cases}$ is asymptotically stable if and only if the coefficient p = -$(a_1 + b_2)$ and q = $a_1b_2 - a_2b_1$ of the auxiliary equation $m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0$ are both positive (4marks)

OR

B (a) Prove that if the critical point(0,0)of $\begin{cases} \frac{dx}{dt} = a_1x + b_1y \\ \frac{dy}{dt} = a_2x + b_2y \end{cases}$ is asymptotically stable then the critical point(0,0) of the non linear system

$$\begin{cases} \frac{dx}{dt} = a_1x + b_1y + f(x,y) \\ \frac{dx}{dt} = a_2x + b_2y + g(x,y) \end{cases}$$ is asymptotically stable (12marks)

(5 × 12 = 60)