KERALA UNIVERSITY

Model Question Paper- M. Sc. Examination

Branch : Mathematics

MM242 - FUNCTIONAL ANALYSIS - II

Time: 3 hours　　　　Max. Marks:75

Part A

Answer any 5 questions from among the questions 1 to 8

Each question carries 3 marks

1. Let X be infinite dimensional normed space and $A \in CL(X)$. Prove that $\sigma_a(A)$ is nonempty.
2. State and prove Schwarz inequality.
3. Let $e_n = (0, 0, \cdots, 0, 1, 0, 0, \cdots)$ where 1 occurs in the n^{th} place. Prove that the series $\sum_n \frac{1}{n} e_n$ is convergent in l^2.
4. Let X be an inner product space and $E \subset X$ be convex. Prove that there exists at most one best approximation from E to any $x \in X$.
5. Let $f : \mathbb{R}^3 \to \mathbb{K}$ be defined by $f(x_1, x_2, x_3) = x_1 + 2x_2 - 3x_3$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the representor of f.
6. Define adjoint of an operator. Find the adjoint of $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A(x_1, x_2) = (x_1 - x_2, 2x_3)$, $(x_1, x_2) \in \mathbb{R}^2$.
7. Let H be a Hilbert space and $A \in BL(H)$. If A is nonzero and self adjoint, prove that A^n is nonzero and self adjoint for any $n \in \mathbb{N}$..
8. Let H be a Hilbert space and $A \in BL(H)$ be normal. Prove that every spectral value of A is an approximate eigen value of A.

$5 \times 3 = 15$

Part B

Answer all questions from 9 to 13

Each question carries 12 marks

9. A a. Let X be a normed linear space and $A \in CL(X)$. Prove that every nonzero spectral value of A is an eigen value of A. 8 marks

   b. Let $A \in CL(X)$ and $k \neq 0$. Prove that $A - kI$ is injective if and only $A - kI$ is surjective. 4 marks

OR

B a. Let X be a normed linear space and $A \in CL(X)$. Prove that the eigen spectrum and spectrum of A are countable sets and zero is the only possible limit point. 5 marks

b. Let $A : l^p \to l^p$ defined by $A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$. Check whether A is compact. 4 marks

c. Prove that every eigen space of a compact operator corresponding to a nonzero eigen value is finite dimensional. 3 marks

10. A. a. Let H be a nonzero Hilbert space. Prove that H has a countable orthonormal basis if and only if H is separable. 8 marks

b. Let X be an inner product space and $x, y \in X$. Prove that x is orthogonal to y if and only if $\|kx + y\|^2 = \|kx\|^2 + \|y\|^2$ for every $k \in K$. 4 marks

OR

B. a. State and prove Bessel's Inequality. 6 marks

b. Let X and Y be inner product spaces. Prove that a linear map $F : X \to Y$ satisfies $\langle F(x), F(y) \rangle = \langle x, y \rangle$ for every $x, y \in X$ if and only if F satisfies $\|F(x)\| = \|x\|$ for every $x \in X$. 6 marks

11. A. a. State and prove Riesz representation theorem. Prove that the Riesz representation theorem may not hold for an incomplete inner product space. 8 marks

b. Let E be a nonempty closed convex subset of a Hilbert space H. Prove that for each $x \in H$, there exists a unique best approximation from E to x. 4 marks

OR

B. a. Prove that every Hilbert space is reflexive. 6 marks

b. Let F be a finite dimensional subspace of an inner product space X. Prove that $X = F + F^{\perp}$. 3 marks

c. Let $F = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + 2x_2 = 0\}$. Find $F^{\perp}$. 3 marks

12. A. a. Let H be a Hilbert space and $A \in BL(H)$. Prove that there exists a unique operator $B \in BL(H)$ such that $\langle Ax, y \rangle = \langle x, By \rangle$ for every $x, y \in H$. 6 marks

b. Let (A_n) be a sequence of operators in $BL(H)$ such that $A_n \to A$ as $n \to \infty$. If each A_n is normal then prove that A is also normal. 3 marks

c. Let $A : l^2 \to l^2$ defined by $A(x_1, x_2, x_3, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$. Check whether A is self adjoint. 3 marks

OR

B. a. Let H be a Hilbert space and $A \in BL(H)$ be self adjoint. Prove that $\|A\| = \sup\{|\langle A(x), x \rangle| : x \in H, \|x\| \leq 1\}$ 6 marks

b. Let $A \in BL(H)$. Prove that $R(A) = H$ if and only if A^* is bounded below. 6 marks

13. A. a. Let $A \in BL(H)$ be a compact operator. Prove that A^* is compact. 6 marks

b. Let H be a Hilbert space and $A \in BL(H)$. Prove that the spectrum of A, $\sigma(A) = \sigma_a(A) \cup \{k : \overline{k} \in \sigma_e(A^*)\}$. 6 marks

OR

B. a. Let H be a Hilbert space and $A \in BL(H)$. Prove that $\sigma_e(A) \subset \omega(A)$ and $\sigma(A)$ is contained in the closure of $\omega(A)$. 6 marks

b. Let A be a compact operator on a nonzero Hilbert space H. If A is self adjoint, then prove that $\|A\|$ or $-\|A\|$ is an eigen value of A. 3 marks

c. Let H be a separable Hilbert space and $A \in BL(H)$. If A is normal, prove that $\sigma_e(A)$ is countable. 3 marks

. $5 \times 12 = 60$