KERALA UNIVERSITY
Model Question Paper- M. Sc. Examination
Branch : Mathematics
MM 244- ADVANCED COMPLEX ANALYSIS

Time: 3 hours Max. Marks: 75

Part A

Answer any 5 questions from among the questions 1 to 8
Each question carries 3 marks

1. If $\{f_n\} \subset H(G)$ converges to$f$ in $H(G)$ and each f_n nevervanishes on G then prove that either $f \equiv 0$ or f never vanishes.
2. State Riemann mapping theorem. Find the equivalent classes among the simply connected regions.
3. Show that $\prod_{n=2}^{\infty}(1-\frac{1}{n^2}) = \frac{1}{2}$
4. If Re $z > 1$, show that $\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty}\int_{0}^{\infty} e^{-nt}t^{z-1}dt$
5. If z is not an integer, prove that $\Gamma(z)\Gamma(1-z) = \pi\ cosec\ \pi z$
6. Define: (i) function element and (ii) analytic continuation along a path.
7. Prove that every harmonic function is infinitely differentiable.
8. Define the order of an analytic function. Also find the order of the analytic function $f(z) = exp(z^4)$. $5 \times 3 = 15$

Part B

Anwer all questions from 9 to 13
Each question carries 12 marks

9. A. *a.* If G is open in $\mathbb{C}$ then prove that there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \cup_{n=1}^{\infty} K_n$, $K_n \subset intK_{n+1}$ and every compact subset of G is a subset of K_n for some n.
   b. If a set $\mathscr{F} \subset C(G, \Omega)$ is normal then prove that:
      i. for each z in G, $\{f(z) : f \in \mathscr{F}\}$ has a compact closure in Ω
      ii. $\mathscr{F}$ is equi continuous at each point of G

   OR

   B. *a.* Prove that a family $\mathscr{F}$ in $H(G)$ is normal if and only if $\mathscr{F}$ is locally bounded
   b. Find an analytic function f which maps $\{z : |z| < 1, Re\ z > 0\}$ on to $B(0, 1)$ which is one-one.

10. A. *a.* If $Re\ z_n > 0$ then prove that Πz_n converges absolutely if and only if the series $\sum(z_n - 1)$ converges absolutely.

b. Show that $\sin \pi z = \pi z \ \Pi_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$

OR

B. *a.* If $G = \{z : Re\ z > 0\}$ and $f_n(z) = \int_{\frac{1}{n}}^{n} e^{-t}t^{z-1}dt$ for $n \geq 1$ and z in G, then prove that each f_n is analytic on G and the sequence is convergent in $H(G)$.

b. If $Re\ z > 0$ then prove that, $\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt$

11. A. *a.* If $Re\ z > 1$, then prove that $\zeta(z)\Gamma(z) = \int_0^{\infty} (e^t - 1)^{-1}t^{z-1}dt$

b. If $a \in \mathbb{C} - K$, then prove that $(z-a)^{-1} \in B(E)$, where E is a subset of $\mathbb{C}_{\infty} - K$

OR

B. Prove the following are equivalent:

i. G is simply connected

ii. Every function f in $H(G)$ has a primitive

iii. G is homeomorphic to the unit disk

12. A. *a.* Let G be a region such that $G = G^*$. If $f : G_+ \cup G_0 \to \mathbb{C}$ is a continuous function which is a nalytic in G_+ and if $f(x)$ is real for x in G_0, then show that there is an analytic function $g : G \to \mathbb{C}$ such that $g(z) = f(z)$ for all z in $G_+ \cup G_0$

b. If $\gamma : [0,1] \to \mathbb{C}$ is a path from a to b and $\{(f_t, D_t) : 0 \leq t \leq 1\}$, $\{(g_t, B_t) : 0 \leq t \leq 1\}$ are analytic continuations along γ such that $[f_0]_a = [g_0]_a$ then prove that $[f_1]_b = [g_1]_b$

OR

B. *a.* If $\gamma : [0,1] \to \mathbb{C}$ is a path and $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ and $R(t)$ is the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$ then show that either $R(t) \equiv \infty$ or $R : [0,1] \to (0, \infty)$ is continuous.

b. If (f, D) is a function element which admits unrestricted continuation in the simply connected region G, show that there is an analytic funcction $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all z in D.

13. A. *a.* Let G be a region and suppose that u is a continuous real valued function on G with the Mean Value Property. If there is a point a in G such that $u(a) \geq u(z)$ for all z in G then prove that u is a constant function.

b. If $D = \{z : |z| < 1\}$ and $f : \partial D \to \mathbb{R}$ is a continuous function then prove that there is a continuous function $u : \overline{D} \to \mathbb{R}$ such that $u(z) = f(z)$ for all z in ∂D and u is harmonic in D

OR

B. *a.* If f is an entire function of genus μ then prove that for each positive number α there is a number r_0 such that $|f(z)| < exp(\alpha|z|^{\mu+1})$ for all $|z| > r_0$.

b. If f is an entire function of finite order λ, where λ is not an integer then prove that f has infinitely many zeros.

. $5 \times 12 = 60$